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Regular functions of biquaternionic variables and Maxwell's equations

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Abstract

In this paper we define a notion of regularity for functions of one and several biquaternionic variables. As a special case we obtain the notion of regularity given by Imaeda (1976) that gives rise to Maxwell's equations. We investigate algebraic and analytic properties of these functions and discuss their physical interpretations.

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1. Introduction

This paper was stimulated by an idea of Imaeda [12] concerning an alternative formulation of electrodynamics based on the study of regular functions of real biquaternionic variables. In his paper Imaeda manipulates the classical Cauchy–Fueter system, obtains a new notion of regularity and shows its relations with Maxwell's equations. He obtains several classical results as a consequence of his regularity condition (e.g. the retarded potential formula and the field generated by a moving charge). In this paper we reformulate his approach in a larger framework, using a new and very natural operator which we introduce in Section 3. This operator allows us to deduce Maxwell's equations and to study their behavior in two Minkowski space–times, one with electric charges and the other with magnetic monopoles.

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Our model, in particular, predicts the lack of wave propagation from one Minkowski space–time to the other, so that the presence of magnetic monopoles in the Minkowski space–time K^{m} is not contradictory to physical reality in the Minkowski space–time K^{e}.

From a mathematical point of view, our work is strongly influenced by our earlier successes with the algebraic treatment of the Cauchy–Fueter system (see, [1,2,4] also [5]). In the present paper we compute the Ext-modules associated with our new operator and provide a physical interpretation for them. In addition, we are able to construct a new hyperfunction theory which can be physically interpreted as a way of describing the behavior of an electromagnetic field as it propagates through a nonlinear material.

After a preliminary section in which the algebra $\mathbb{BH}$ of biquaternions is defined, we introduce a new operator and the corresponding regularity conditions. Section 4 is devoted to the algebraic study of this and related operators, and to the analytic consequences of our results. In Section 5 we study the analysis of regular functions which is crucial for the construction of regular hyperfunctions in Section 7. In Section 6, on the other hand, we provide the physical interpretation for the new operator and regular functions.

2. Preliminaries

In this section we define the algebra $\mathbb{BH}$ of biquaternions and discuss its elementary algebraic properties. The associative complex algebra of biquaternions $\mathbb{BH}$ is defined as the complex algebra generated over the basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_0 = 1$ and $\mathbf{e}_k$, $k = 1, 2, 3$, satisfy the following Pauli-type algebraic relations

$$\mathbf{e}_k^2 = 1, \quad k = 1, 2, 3, \quad \mathbf{e}_\ell \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_\ell = \mathrm{i}\mathbf{e}_k, \quad \mathrm{i} = \sqrt{-1} \in \mathbb{C}$$

with l, j, k being any cyclic permutation of 1, 2, 3. The relations among the units $\mathbf{e}_k$, $k = 1, 2, 3$, are the same as the ones among the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

According to these definitions, a biquaternion Z is an element

$$Z = \mathbf{e}_0 z_0 + \mathbf{e}_1 z_1 + \mathbf{e}_2 z_2 + \mathbf{e}_3 z_3,$$

where z_μ are complex numbers written as $z_\mu = x_\mu + \mathrm{i}y_\mu$, $\mu = 0, 1, 2, 3$. We write a biquaternion as

$$Z = x_0 + \mathbf{x} + \mathrm{i}y_0 + \mathrm{i}\mathbf{y}$$

with $\mathbf{x} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3$ and $\mathbf{y} = \mathbf{e}_1 y_1 + \mathbf{e}_2 y_2 + \mathbf{e}_3 y_3$. We can define the so-called hyperconjugate Z^+

$$Z^+ = x_0 - \mathbf{x} + \mathrm{i}y_0 - \mathrm{i}\mathbf{y}. \tag{2.1}$$

We say that a biquaternion X is real if it has zero imaginary part, i.e. $X = x_0 + \mathbf{x}$. The subset of real biquaternions is denoted by $\mathbb{RH}$ and is called (see [12]) "real biquaternion space". Every biquaternion Z in $\mathbb{BH}$ can be written as $Z = X + \mathrm{i}X'$, with $X, X' \in \mathbb{RH}$. The norm of a biquaternion Z, defined as

$$N(Z) = Z^+Z = ZZ^+ = z_0^2 - z_1^2 - z_2^2 - z_3^2, \tag{2.2}$$

is, in general, a complex number. This norm becomes real if and only if the components (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) of X and X' are orthogonal with respect to the Minkowski space–time inner product, i.e. $\langle X, g_{\mu\nu}X'\rangle = 0$, where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. When we restrict our attention to a real biquaternion X, we have $N(X) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \in \mathbb{R}$ and, as Imaeda points out in [12], the metric space structure of the space of real biquaternions is equal to that of a Minkowski space. When the norm $N(Z)$ of Z is zero, but $Z \neq 0$, we say that Z is a zero divisor. It is important to note that the biquaternion algebra, unlike the real quaternion algebra, is not a division algebra because of the existence of zero divisors. Given a biquaternion Z such that $N(Z) \neq 0$ we can define its inverse as

$$Z^{-1} = \frac{Z^+}{N(Z)}.$$

The set $\mathbb{BH}$, considered as a ring, contains the ring $\mathbb{H}$ of quaternions, as a subring. A quaternion q is written in this case as $q = \mathbf{e}_0x_0 + \mathrm{i}\mathbf{e}_1y_1 + \mathrm{i}\mathbf{e}_2y_2 + \mathrm{i}\mathbf{e}_3y_3$, where the units $\mathbf{e}_\mu$, $\mu = 0, 1, 2, 3$, are related to the units of $\mathbb{H}$ by

$$\mathrm{i}\mathbf{e}_1 = \mathbf{i}, \quad \mathrm{i}\mathbf{e}_2 = \mathbf{j}, \quad \mathrm{i}\mathbf{e}_3 = \mathbf{k}. \tag{2.3}$$

The set of units $\{\mathbf{e}_0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a basis of $\mathbb{H}$ and satisfies the following multiplication relations (note the difference in sign from the usual quaternionic variables):

$$\mathbf{ji} = -\mathbf{ij} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{ki} = \mathbf{j}, \qquad \mathbf{kj} = -\mathbf{kj} = \mathbf{i}.$$

We are aware that this notation differs from the standard one. In fact, it is possible to relate the units $\mathbf{e}_\mu$ with the units of $\mathbb{H}$ in many ways. However, we decided to use this particular choice for two reasons. First, it follows Imaeda's [12]; second, this particular choice allows us to obtain the traditional Cauchy–Fueter system directly from our matrix.

We note that $\mathbb{BH}$ is isomorphic, as a $\mathbb{C}$-vector space, to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, the complexified quaternions.

3. Regularity conditions

It is well known that it is possible to define a notion of (left) regularity for functions $f : U \subset \mathbb{H} \longrightarrow \mathbb{H}$ of class $\mathcal{C}^1$. Regular functions are defined as the kernel of the Cauchy–Fueter operator

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial y_1} + \mathbf{j}\frac{\partial}{\partial y_2} + \mathbf{k}\frac{\partial}{\partial y_3}. \tag{3.1}$$

In [12] a similar condition is given to characterize regular functions of a real biquaternion variable. In order to do so, Imaeda formally replaces y_k by $-\mathrm{i}x_k$ in (3.1) to obtain the operator

$$D = \frac{\partial}{\partial x_0} - \mathbf{e}_1\frac{\partial}{\partial x_1} - \mathbf{e}_2\frac{\partial}{\partial x_2} - \mathbf{e}_3\frac{\partial}{\partial x_3}. \tag{3.2}$$

A function $F : \mathbb{RH} \longrightarrow \mathbb{BH}$,

$$F(X) = \sum_{\nu=0}^{3}(a_\nu(x_\mu) + \mathrm{i}b_\nu(x_\mu))\mathbf{e}_\nu,$$

where $a_\nu(x_\mu)$, $b_\nu(x_\mu)$ are real-valued functions, is said to be (left) D-regular if it is of class $\mathcal{C}^1$ and satisfies

$$DF(X) = 0. \tag{3.3}$$

If we put $(a_0, \mathbf{a}) = (a_0, a_1, a_2, a_3)$ and $(b_0, \mathbf{b}) = (b_0, b_1, b_2, b_3)$, the regularity condition in vector notation can be written as

$$\begin{aligned} &\frac{\partial}{\partial x_0}a_0 - \operatorname{div}\mathbf{a} = 0, \qquad \frac{\partial}{\partial x_0}b_0 - \operatorname{div}\mathbf{b} = 0, \\ &\frac{\partial}{\partial x_0}\mathbf{a} - \operatorname{grad} a_0 + \operatorname{curl}\mathbf{b} = 0, \qquad \frac{\partial}{\partial x_0}\mathbf{b} - \operatorname{grad} b_0 - \operatorname{curl}\mathbf{a} = 0. \end{aligned} \tag{3.4}$$

This system represents Maxwell's equations (see [12]) if the vectors **a** and **b** represent the magnetic and the electric fields, respectively; b_0 is related to the electric density charge $\rho^{\mathrm{e}}(X)$ and to the electric current density J^{e} by the relations $\rho^{\mathrm{e}}(X) = \partial/\partial x_0 b_0$, $J^{\mathrm{e}} = -\operatorname{grad} b_0$. Moreover, we assume that the scalar a_0 is a constant to avoid the existence of magnetic monopoles.

If we think of $F(X)$ as an 8-vector, we can write system (3.3) in the following matrix form:

$$\begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} & 0 & 0 & 0 & 0 \\ -\partial_{x_1} & \partial_{x_0} & 0 & 0 & 0 & 0 & -\partial_{x_3} & \partial_{x_2} \\ -\partial_{x_2} & 0 & \partial_{x_0} & 0 & 0 & \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_3} & 0 & 0 & \partial_{x_0} & 0 & -\partial_{x_2} & \partial_{x_1} & 0 \\ 0 & 0 & 0 & 0 & \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ 0 & 0 & \partial_{x_3} & -\partial_{x_2} & -\partial_{x_1} & \partial_{x_0} & 0 & 0 \\ 0 & -\partial_{x_3} & 0 & \partial_{x_1} & -\partial_{x_2} & 0 & \partial_{x_0} & 0 \\ 0 & \partial_{x_2} & -\partial_{x_1} & 0 & -\partial_{x_3} & 0 & 0 & \partial_{x_0} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0. \tag{3.5}$$

In the sequel it will be useful to write the matrix associated to system (3.5) as follows:

$$\mathcal{M}_D = \begin{bmatrix} A & C \\ -C & A \end{bmatrix}, \tag{3.6}$$

where

$$A = A^{\mathrm{t}} = \begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ -\partial_{x_1} & \partial_{x_0} & 0 & 0 \\ -\partial_{x_2} & 0 & \partial_{x_0} & 0 \\ -\partial_{x_3} & 0 & 0 & \partial_{x_0} \end{bmatrix}$$

and

$$C = -C^{\mathrm{t}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_{x_3} & \partial_{x_2} \\ 0 & \partial_{x_3} & 0 & -\partial_{x_1} \\ 0 & -\partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix}.$$

We now want to extend the operator defined by the 8×8 matrix $\mathcal{M}_D$ to an operator acting not only on functions of a real biquaternion variable, but on functions $F : \mathbb{BH} \longrightarrow \mathbb{BH}$ defined on all $\mathbb{BH}$.

We now define the operator

$$\mathcal{T} = \frac{\partial}{\partial Z^+} = \frac{\partial}{\partial z_0} - \sum_{j=1}^{3} \mathbf{e_j} \frac{\partial}{\partial z_j}, \tag{3.7}$$

where

$$\frac{\partial}{\partial z_\mu} = \frac{\partial}{\partial x_\mu} - \mathrm{i}\frac{\partial}{\partial y_\mu}, \qquad \mu = 0, 1, 2, 3$$

and Z^+ is defined in (2.1). In view of the above considerations it is clear that $\mathcal{T}$ is a generalization of the Cauchy–Riemann operator $\partial/\partial\bar{z}$ for functions of one complex variable or (as explained above) of the Cauchy–Fueter operator $\partial/\partial\overline{q}$.

Where the more explicit notation $\partial/\partial Z^+$ is needed, we will use it instead of $\mathcal{T}$.

Definition 3.1. A function $F : U \subseteq \mathbb{BH} \longrightarrow \mathbb{BH}$ of class $\mathcal{C}^1$ on the open set U is said to be left $\mathcal{T}$-regular if

$$\mathcal{T}F = 0$$

and right $\mathcal{T}$-regular if

$$F\mathcal{T} = 0,$$

where $F\mathcal{T} = 0$ is defined by

$$F\mathcal{T} = \frac{\partial F}{\partial z_0} - \sum_{j=1}^{3} \frac{\partial F}{\partial z_j}\mathbf{e_j} = 0.$$

It is important to verify that the matrix $\mathcal{M}_\mathcal{T}$ associated to operator $\mathcal{T}$ is

$$\mathcal{M}_\mathcal{T} = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix}, \tag{3.8}$$

where

$$P = P^{\mathrm{t}} = \begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ -\partial_{x_1} & \partial_{x_0} & \partial_{y_3} & -\partial_{y_2} \\ -\partial_{x_2} & -\partial_{y_3} & \partial_{x_0} & \partial_{y_1} \\ -\partial_{x_3} & \partial_{y_2} & -\partial_{y_1} & \partial_{x_0} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \partial_{y_0} & -\partial_{y_1} & -\partial_{y_2} & -\partial_{y_3} \\ -\partial_{y_1} & \partial_{y_0} & -\partial_{x_3} & \partial_{x_2} \\ -\partial_{y_2} & \partial_{x_3} & \partial_{y_0} & -\partial_{x_1} \\ -\partial_{y_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{y_0} \end{bmatrix},$$

and its Fourier transform is the matrix

$$\hat{\mathcal{M}}_{\mathcal{T}} = \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 & y_0 & -y_1 & -y_2 & -y_3 \\ -x_1 & x_0 & y_3 & -y_2 & -y_1 & y_0 & -x_3 & x_2 \\ -x_2 & -y_3 & x_0 & y_1 & -y_2 & x_3 & y_0 & -x_1 \\ -x_3 & y_2 & -y_1 & x_0 & -y_3 & -x_2 & x_1 & y_0 \\ -y_0 & y_1 & y_2 & y_3 & x_0 & -x_1 & -x_2 & -x_3 \\ y_1 & -y_0 & x_3 & -x_2 & -x_1 & x_0 & y_3 & -y_2 \\ y_2 & -x_3 & -y_0 & x_1 & -x_2 & -y_3 & x_0 & y_1 \\ y_3 & x_2 & -x_1 & -y_0 & -x_3 & y_2 & -y_1 & x_0 \end{bmatrix}.$$

Remark 3.2.

Both Maxwell's equations and the Cauchy–Fueter equations are now particular cases of $\mathcal{T}\mathcal{F} = 0$ when we restrict the domain and/or the range of the function F. Indeed, if we consider $F : \mathbb{RH} \longrightarrow \mathbb{BH}$, the operator $\mathcal{T}$ characterizes D-regular functions, and hence leads to Maxwell's equations, while, if we consider $F : \mathbb{H} \subset \mathbb{BH} \longrightarrow \mathbb{H} \subset \mathbb{BH}$ we obtain the usual quaternionic regular functions and hence the Cauchy–Fueter equations. Another interesting feature of the operator $\mathcal{T}$ is that it also contains the conditions of regularity for functions of two quaternionic variables. In fact, we can split a biquaternion as the sum of two quaternions $Z = q + \mathrm{i}q'$, where $q = x_0 + \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3$, $q' = y_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$ and we can think of a function F defined on $\mathbb{BH}$ as $F = F(q, q')$. If we consider a function $F : \mathbb{BH} \longrightarrow \mathbb{H}$, where $F = F_0 + \mathbf{i}F_1 + \mathbf{j}F_2 + \mathbf{k}F_3$, and impose $\mathcal{T}\mathcal{F} = 0$, we will obtain the system

$$\frac{\partial f}{\partial \overline{q}} = 0, \quad \frac{\partial f}{\partial \overline{q}'} = 0,$$

which corresponds to the Cauchy–Fueter system for functions of two quaternionic variables.

4. Algebraic results

In this section we discuss the algebraic properties of the operators D, $\mathcal{T}$ and some of their variations with the methods which we have introduced in our previous papers [1,2,4]. The idea is to use such methods to study compact and pointwise singularities for such systems, as well as to build a new version of the hyperfunctions theory which has some interesting interpretations in terms of electromagnetic fields. As we have shown in [1] (and related papers), a great amount of analytical information is contained in the complex of syzygies of the module associated to the system, and therefore we will begin this section with its study.

Throughout this section, A will be either the ring $A = \mathbb{R}[x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]$ of polynomials in the eight real variables $x_0, \ldots, y_3$ or the ring $A = \mathbb{R}[x_0, x_1, x_2, x_3]$ of polynomials in the real variables $x_0, \ldots, x_3$. The context will make it clear to the reader which case is being used.

Our first goal will be to compute the Ext-modules associated to the module M generated by the matrix $\hat{\mathcal{M}}_{\mathcal{T}}$, i.e.

$$M = A^8/\hat{\mathcal{M}}_{\mathcal{T}} A^8.$$

It is possible to compute, using CoCoA, the syzygy module [1] of the columns of the matrix $\hat{\mathcal{M}}_{\mathcal{T}}$. This syzygy module is equal to zero, i.e. $\mathrm{Ext}^0(M, A) = 0$, which implies, (see [13,15]), the unique continuation property for the solutions of the system $\mathcal{T}F = 0$. We now show that $\mathrm{Ext}^1(M, A) \neq 0$. Since the matrix $\hat{\mathcal{M}}_{\mathcal{T}}$ has maximal rank and is square, $\mathrm{Ext}^1(M, A) = 0$ if and only if the determinant of $\hat{\mathcal{M}}_{\mathcal{T}}$ is 1, see [1]. The determinant of $\hat{\mathcal{M}}_{\mathcal{T}}$ can be computed to be

$$f = [(N(X) - N(X'))^2 + 4\langle X, g_{\mu\nu} X'\rangle^2]^2 = |N(Z)^2|^2$$

from which we conclude that $\mathrm{Ext}^1(M, A) \neq 0$. This result, while not surprising (a similar phenomenon occurs in the complex and quaternionic cases), can be given the analytic interpretation that $\mathcal{T}$-regular functions can have compact singularities (see [15]). We will show later how this result can be modified. In view of Palamodov's results on the characteristic varieties of the Ext-modules, it is important for us to determine the characteristic variety of $\mathrm{Ext}^1(M, A)$. In the case in hand, we can use the following general result.

Theorem 4.1. *Let $A^s \xrightarrow{A} A^q$ be a linear transformation, where $q \geq s$, with A of maximal rank s and let $M = A^q/AA^s$. Then the characteristic variety of $\mathrm{Ext}^1(M, A)$, as a set, is the variety defined by the greatest common divisor d of all the $s \times s$ minors of A.*

Proof. First, it is well known that $\mathrm{Ext}^1(M, A)$ is the torsion submodule N of A^q/AA^s. Let

$$F_1 \xrightarrow{K} F_0 \longrightarrow N \longrightarrow 0$$

be a free resolution of this module. The characteristic variety of N is defined by the ideal F_0 generated by the maximal minors of K. This ideal is also known as the 0th Fitting ideal

[1] The algorithm for the computation of the syzygy module is based on the theory of Gröbner bases (see [3]). Once the ring A and the module M are defined in CoCoA, the command `Syz(M)` gives the syzygies of M.

$\mathcal{G}_0$ of N. It is also well known (see, for example, [10]) that for some n,

$$(\mathrm{ann}(N))^n \subseteq \mathcal{G}_0 \subseteq \mathrm{ann}(N),$$

where $\mathrm{ann}(N) \subseteq A$ denotes the annihilator of N. Therefore, the varieties defined by $\mathrm{ann}(N)$ and by $\mathcal{G}_0$ are the same as sets. We claim that the only prime ideals above $\mathrm{ann}(N)$ are the ideals generated by prime divisors of d. Indeed, if $d \notin P$, P a prime ideal, then the map

$$A^s \otimes_A A_P \xrightarrow{A} A^q \otimes_A A_P$$

still has maximal rank, where A_P denotes the localization of A at P. But in this situation, the gcd of the $s \times s$ minors is 1 (since d in A_P is invertible). So A^q_P / AA^s_P has no torsion elements (see [1]). So P cannot be above the annihilator of N and hence, as a set, the characteristic variety is defined by d. □

We now apply this result to the system $\mathcal{M}_\mathcal{T}$ to obtain that the characteristic variety of $\mathrm{Ext}^1(M, A)$ for $\mathcal{M}_\mathcal{T}$ is the variety defined by the polynomial f above (this could have been easily computed in this case without Theorem 4.1, since the syzygy module of $\hat{\mathcal{M}}^t_\mathcal{T}$ is zero, so $\mathrm{Ext}^1(M, A) \cong A^q / \hat{\mathcal{M}}^t_\mathcal{T} A^s$). The full power of Theorem 4.1 will be used later.

If we consider the matrix $\mathcal{M}_D$, then again it is easy to compute that $\mathrm{Ext}^1(M, A) \neq 0$, and the characteristic variety is defined by the polynomial

$$g = (x_0^2 - x_1^2 - x_2^2 - x_3^2)^4 = N(X)^4,$$

i.e. the characteristic variety is, geometrically, the light cone in the dual space. We will see in the sequel how this result can be utilized.

The fact that $\mathrm{Ext}^1(M, A) \neq 0$ means that the system is not overdetermined and, therefore, compact singularities cannot be eliminated. Given the physical meaning of the system $\mathcal{M}_D$, the possible existence of compact singularities may be interpreted as the presence of localized charges which generate the electromagnetic field. One may be tempted to change this situation by looking for special situations in which the field **a** and **b** are proportional (plane waves), or even **a** = **b** (as will be shown in Section 5, one can interpret **a** as the magnetic field and **b** as the electric field). This suggests the study of a new system

$$(A + C)\mathbf{a} = 0, \quad (A - C)\mathbf{a} = 0$$

associated to an 8×4 matrix whose Fourier transform is

$$\begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ -x_1 & x_0 & -x_3 & x_2 \\ -x_2 & x_3 & x_0 & -x_1 \\ -x_3 & -x_2 & x_1 & x_0 \\ x_0 & -x_1 & -x_2 & -x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{bmatrix}.$$

On the basis of some earlier results [2], we might expect overdeterminacy but, in fact, a quick computation shows that, once again, $\mathrm{Ext}^1(M, \Lambda) \neq 0$ (i.e. the gcd of the 4×4 minors is not 1). Physically this may be interpreted by saying that, in the presence of an electromagnetic field, electric charges must exist. Once again, the obstruction to the vanishing of $\mathrm{Ext}^1(M, \Lambda)$ is given by the light cone and the characteristic variety is

$$\{X: N(X) = x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0\}.$$

The fact that we have a complete description of the characteristic variety has a rather remarkable and suprising consequence. Indeed, we have the following result:

Theorem 4.2. *Let Ω be an open set in $\mathbb{BH}$ and let $P \in \Omega$. Then every $\mathcal{T}$-regular function on $\Omega \setminus \{P\}$ whose components extend as distributions to all of Ω is, indeed, a distribution solution to the system $\mathcal{T}$ on all Ω.*

Proof. This result follows immediately from Corollary 8.14.4 in [15], in view of the fact that the characteristic variety of $\mathrm{Ext}^1(M, \Lambda)$ is the light cone which is, obviously, non-hypoelliptic (see [15]). □

Remark 4.3. Since the system $\mathcal{T}$ is not elliptic, distributions solutions are not necessarily $\mathcal{T}$-regular functions (unlike what happens for the classical Cauchy–Fueter system).

Remark 4.4. It is important to notice, once again, that this phenomenon is quite new, as one-point singularities do occur for regular functions (in the Cauchy–Riemann and Cauchy–Fueter sense) in one variable.

Remark 4.5. A completely analogous result can be formulated for D-regular functions.

We now turn our attention to what happens for functions of several biquaternionic variables. For the sake of simplicity we will restrict our attention to the case of a function $F: (\mathbb{BH})^2 \longrightarrow \mathbb{BH}$ of two biquaternionic variables Z and W. In this case, we will say that F is $\mathcal{T}$-regular if

$$\frac{\partial F}{\partial Z^+} = \frac{\partial F}{\partial W^+} = 0.$$

As we did in [1], we can use CoCoA to compute the resolution of the module [2] associated to the 16×8 matrix which describes the above system. Without giving any computational details, we have the following theorem.

Theorem 4.6. *The module M associated to the system*

$$\frac{\partial F}{\partial Z^+} = 0, \quad \frac{\partial F}{\partial W^+} = 0$$

[2] The algorithm for the computation of the resolution of a module is based on the theory of Gröbner bases (see [3]). Once the ring Λ and the module M are defined in CoCoA, the command `Res(M)` gives the resolution of M, which is minimal if M is homogeneous.

admits a resolution of length 4

$$0 \longrightarrow \Lambda^8 \longrightarrow \Lambda^{16} \longrightarrow \Lambda^{16} \longrightarrow \Lambda^8 \longrightarrow 0.$$

We also have

$$Ext^0(M, \Lambda) = Ext^1(M, \Lambda) = Ext^2(M, \Lambda) = 0,$$

while $Ext^3(M, \Lambda) \neq 0$. *Therefore removability of compact singularities occurs.*

Here Λ is the ring of polynomials in 16 real variables, eight for each biquaternionic variable.

5. Some properties of $\mathcal{T}$-regular functions

In this section we prove the extension of some classical theorems from the theory of holomorphic functions, to the case of $\mathcal{T}$-regular functions.

The set of left $\mathcal{T}$-regular functions on an open set $U \subseteq \mathbb{BH}$ will be denoted by $\mathcal{R}_l^{\mathcal{T}}(U)$ while the set of right $\mathcal{T}$-regular functions will be denoted by $\mathcal{R}_r^{\mathcal{T}}(U)$. If no confusion arises, we will omit the indices l or r.

The following fact can be immediately verified and it is, in fact, a natural property shared by all "regular" functions in Clifford analysis.

Proposition 5.1. $\mathcal{R}_l^{\mathcal{T}}(U)$ *is a right* $\mathbb{BH}$*-module.*

Remark 5.2. It is well known that, in general, (see [18]), regular functions on a non-division algebra are not harmonic. In fact, in our case we have

$$\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^+} = \frac{\partial}{\partial Z^+}\frac{\partial}{\partial Z} = \frac{\partial^2}{\partial z_0^2} - \sum_{j=1}^{3}\frac{\partial^2}{\partial z_j^2}$$

$$= \Delta_y - \Delta_x + \left(\frac{\partial}{\partial x_0} + \mathrm{i}\frac{\partial}{\partial y_0}\right)^2 - 2\mathrm{i}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j \partial y_j}, \tag{5.1}$$

which is an ultra-hyperbolic operator. We will see in the next section the physical meaning of this operator.

Lemma 5.3 (Poincaré lemma). *Let* $U \subseteq \mathbb{BH}$ *be a convex open set, and let* $g : U \longrightarrow \mathbb{BH}$ *be a* $\mathcal{C}^\infty$ *function. Then there is* $f \in \mathcal{C}^\infty(U)$ *such that* $\partial/\partial Z^+ f = g$ *on* U.

Proof. The computation of the syzygy module of the columns of the matrix $\mathcal{T}$ was done in Section 4, and shows that the matrix of the compatibility conditions g is the null matrix. Our result then follows from a standard result of Ehrenpreis [9]. □

We now introduce some differential forms aimed at the formulation of a Cauchy-type integral formula. Let

$$DZ = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 + \mathbf{e}_1\, \mathrm{d}z_0 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 + \mathbf{e}_2\, \mathrm{d}z_0 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_1 + \mathbf{e}_3\, \mathrm{d}z_0 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2,$$
$$\nu = \mathrm{d}z_0 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3.$$

We have the following immediate generalization of the quaternionic case (see [19]).

Proposition 5.4. $\mathrm{d}(g\, DZ\, f) = \mathrm{d}g \wedge DZ\, f - g\, DZ \wedge \mathrm{d}f = \{(g\mathcal{T})f + g(\mathcal{T}f)\}\nu$ *on* $\mathbb{BH}$.

Corollary 5.5. *If f is left $\mathcal{T}$-regular and g is right $\mathcal{T}$-regular, we obtain* $\mathrm{d}(g\, DZ\, f) = 0$.

Corollary 5.6. *A real differentiable function f is $\mathcal{T}$-regular at the point Z if and only if*

$$DZ \wedge \mathrm{d}f = 0$$

at the point Z.

From this last property and Stokes' theorem, we can obtain the Cauchy formula for $\mathcal{T}$-regular functions. It is, first, necessary to introduce the following function:

$$G(Z) = \frac{Z^+}{N(Z)^2},$$

where $N(Z)$ is defined in (2.2). Note that G is the fundamental solution for $\partial/\partial Z^+$, and easy calculations show that the following proposition holds.

Proposition 5.7. *$G(Z)$ is left and right regular on $\mathbb{BH} \setminus \{N(Z) = 0\}$.*

Theorem 5.8 (Cauchy I). *Let $U \subset \mathbb{BH}$ be an open set, and let Σ be a compact 3-chain, boundary of a 4-chain S in U. Then, if g is right $\mathcal{T}$-regular and f is left $\mathcal{T}$-regular, then*

$$\int_{\Sigma} g\, DZ\, f = 0.$$

Proof. Stokes' theorem gives

$$\int_{\Sigma} g\, DZ\, f = \int_{S} \mathrm{d}(g\, DZ\, f) = 0. \qquad \square$$

It is known that, in general, the Cauchy formula holds without limitations in any real Clifford algebra. However in complexified Clifford algebras that are not division algebras, the Cauchy kernel at point P is not necessarily defined, and in our case, it is defined only outside the translated light cone, i.e. in $\mathbb{BH} \setminus \{N(Z - Z_P) = 0\}$. Moreover, as pointed out in [7] (where some integral formulas in hypercomplex analysis are given), if a contour of integration Σ is homologically trivial in an open set Ω contained in a complexified Clifford algebra, it is not necessarily true that Σ is homologically equivalent to a sphere around P.

Let us indicate by $\mathbb{C}N_P$ the set $\{Z \in \mathbb{BH} : N(Z - Z_P) = 0\}$, i.e. $\mathbb{C}N_P$ is the light cone with vertex in P.

Remark 5.9. Identifying $\mathbb{BH}$ with the space of 2×2 matrices with complex coefficients, we see that $N(Z)$ is the determinant and therefore one can identify $\mathbb{BH} \setminus \mathbb{C}N_0$ with $GL(2, \mathbb{C})$. This has the homotopy type of the maximal compact subgroup $U(2)$, and the nontrivial cohomology of this is $\mathbb{Z}$ only in dimensions 0, 1, 3, 4 (we are indebted to the anonymous referee for this remark).

Following [7] it is possible to compute explicitly a generator for the group

$$H_3(\mathbb{BH} \setminus \mathbb{C}N_P, \mathbb{Z})$$

and so to characterize the cycles that can be used to write the Cauchy formula.

Theorem 5.10. *A generator for the group $H_3(\mathbb{BH} \setminus \mathbb{C}N_P, \mathbb{Z}) \cong \mathbb{Z}$ is the sphere*

$$S^3 = \left\{ Z_P + Z : Z = x_0 + \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3 \in \mathbb{H},\ x_0^2 + \sum_{i=0}^{3} y_i^2 = 1 \right\}.$$

Proof. Without loss of generality, we can assume that $P = O$. Obviously, there exists an inclusion $i : S^3 \longrightarrow \mathbb{BH} \setminus \mathbb{C}N_0$, so it suffices to show that the morphism

$$i^* : H_3(S^3, \mathbb{Z}) \longrightarrow H_3(\mathbb{BH} \setminus \mathbb{C}N_0, \mathbb{Z})$$

induced by i is an isomorphism. Let us define the hypersurface S_7 as

$$S_7 = \left\{ Z \in \mathbb{BH}: \sum_{i=0}^{3} (x_i^2 + y_i^2) = 1 \right\}.$$

Let $E = S_7 \cap (\mathbb{BH} \setminus \mathbb{C}N_0)$ and let $p : E \longrightarrow S^3$ be the fibration

$$Z \in E \longrightarrow \frac{x_0^2 + \sum_{i=1}^{3} y_i^2}{\left|\sum_{i=0}^{3} z_i^2\right|}.$$

Let us now consider $F_0 = p^{-1}(1)$. We have

$$F_0 = \left\{ Z \in \mathbb{BH}: \sum_{i=0}^{3} (x_i^2 + y_i^2) = 1,\ x_0 y_0 - \sum_{i=1}^{3} x_i y_i = 0, \right.$$
$$\left. (x_0^2 - y_0^2) - \sum_{i=1}^{3} (x_i^2 - y_i^2) > 0 \right\}.$$

We obtain the following inclusions:

$$S^3 \overset{i''}{\hookrightarrow} F_0 \overset{i'}{\hookrightarrow} E \overset{i}{\hookrightarrow} \mathbb{BH} \setminus \mathbb{C}N_0.$$

To prove that i' and i induce isomorphisms at the homology level can be done exactly as in [7]. In fact, E is a deformation retract of $\mathbb{BH} \setminus \mathbb{C}N_0$ so i'^* is an isomorphism, while the Wang sequence (see [14]) assures us that i^* is an isomorphism.

Finally it is easy to show that S^3 is a deformation retract of F_0. □

Definition 5.11. A domain $\Omega \subset \mathbb{BH}$ is said to be *null-convex* if for all $Z, Z' \in \Omega$ such that $N(Z - Z') = 0$, the whole segment ZZ' belongs to Ω.

Note that if Ω is null-convex, then a cycle Σ can be deformed to a sphere around a point P near the cone $\mathbb{C}N_P$.

Theorem 5.12 (Cauchy II). *Let $\Omega \subset \mathbb{BH}$ be a null-convex domain and $f \in \mathcal{R}^{\mathcal{T}}(\Omega)$. If $P \in \Omega$, then*

$$f(P)\,\mathrm{Ind}_{\Sigma}(P) = \frac{1}{2\pi^2}\int_{\Sigma} G(Z-P)\,DZ\,f(Z),$$

where $\Sigma \subset \Omega$ is any cycle homologous to the 3-sphere S^3.

Proof. We prove the theorem in the case $P = O$. It is obvious that the theorem holds for $\Sigma = S^3$. In fact, it suffices to repeat the arguments given in [19] since S^3 is a sphere in $\mathbb{H}$. In the general case, it suffices to use the fact that $\Sigma \sim nS_3$. □

To have a theory of $\mathcal{T}$-regular functions that parallels one of regular functions of a quaternionic variable, it is also necessary to write Taylor series expansions for $\mathcal{T}$-regular functions. We remark that the operator $\mathcal{T}$ can be decomposed into two operators as follows:

$$\mathcal{T} = D_{x_0,\mathbf{x}} + \mathrm{i}D_{y_0,\mathbf{y}},$$

where

$$D_{x_0,\mathbf{x}} := \frac{\partial}{\partial x_0} - \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial}{\partial x_j} \quad \text{and} \quad D_{y_0,\mathbf{y}} := \frac{\partial}{\partial y_0} - \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial}{\partial y_j}.$$

In [12] a Taylor series is given for D-regular functions of a biquaternion variable. Let

$$P_\nu(Z) = \frac{1}{n!} \sum_{1\le k_1,\dots,k_n\le 3} (z_{k_1} + \mathbf{e}_{k_1} z_0)\cdots(z_{k_n} + \mathbf{e}_{k_n} z_0),$$

where $\nu = (n_1, n_2, n_3)$ such that $n_1 + n_2 + n_3 = n$; the sum is taken over all different orderings of n_1 1's, n_2 2's, n_3 3's. Let us denote by σ_n the set of $[n_1, n_2, n_3]$ such that $n_1 + n_2 + n_3 = n$. If f is a D-regular function, then

$$F(Z) = \sum_{n=0}^{+\infty} \sum_{\nu\in\sigma_n} a_\nu P_\nu(Z), \quad a_\nu \in \mathbb{BH}. \tag{5.2}$$

Let us remark that $P_\nu(Z)$ are not only D-regular, as Imaeda pointed out (i.e. $D_{x_0,\mathbf{x}}$-regular, with our notation) but also $D_{y_0,\mathbf{y}}$-regular because the x variables and y variables play a symmetric role in the polynomials P_ν. So expansion (5.2) holds, not only for D-regular functions (with respect to the variables x or y), but also for $\mathcal{T}$-regular functions.

Remark 5.13. The expansion (5.2) implies that a regular function is a real infinitely differentiable function.

6. Some physical comments

The importance of quaternions and, more in general, of Clifford algebras in different fields of theoretical physics is well known. To have an overview of recent progress, we refer the reader, for example, to the references contained in [6]

In this section we provide a possible physical interpretation of the kernel of the operator $\mathcal{T}$.

If we explicitly write, in vector notation, the regularity condition $\mathcal{T}F = 0$, we get the following system:

$$\begin{aligned}
&\frac{\partial}{\partial x_0}a_0 + \frac{\partial}{\partial y_0}b_0 - \mathrm{div}_x\,\mathbf{a} - \mathrm{div}_y\,\mathbf{b} = 0,\\
&\frac{\partial}{\partial y_0}a_0 - \frac{\partial}{\partial x_0}b_0 - \mathrm{div}_y\,\mathbf{a} + \mathrm{div}_x\,\mathbf{b} = 0,\\
&\frac{\partial}{\partial x_0}\mathbf{a} + \frac{\partial}{\partial y_0}\mathbf{b} - \mathrm{grad}_x\,a_0 - \mathrm{grad}_y\,b_0 + \mathrm{curl}_x\,\mathbf{b} - \mathrm{curl}_y\,\mathbf{a} = 0,\\
&\frac{\partial}{\partial x_0}\mathbf{b} - \frac{\partial}{\partial y_0}\mathbf{a} - \mathrm{grad}_x\,b_0 + \mathrm{grad}_y\,a_0 - \mathrm{curl}_x\,\mathbf{a} - \mathrm{curl}_y\,\mathbf{b} = 0.
\end{aligned} \tag{6.1}$$

Let K^{e} and K^{m} be Minkowski space–times with coordinates $(x_0, \mathbf{x})$ and $(y_0, \mathbf{y})$, respectively. The variables x_0 and y_0 represent the time coordinates while $\mathbf{x}$ and $\mathbf{y}$ represent the spatial variables. We assume that in K^{e} there are only electric monopoles and in K^{m} there are only magnetic monopoles. If this holds, then the terms a_0 and b_0 depend only on some time and spatial variables, because

$$\begin{aligned}
&\frac{\partial}{\partial y_0}a_0 := \rho^{\mathrm{m}}(y_0, \mathbf{y}) \text{ is the magnetic monopole density in } K^{\mathrm{m}},\\
&\frac{\partial}{\partial x_0}b_0 := \rho^{\mathrm{e}}(x_0, \mathbf{x}) \text{ is the electric monopole density in } K^{\mathrm{e}},\\
&\mathrm{grad}_y\,a_0 := J^{\mathrm{m}}(y_0, \mathbf{y}) \text{ is the magnetic current density in } K^{\mathrm{m}},\\
&\mathrm{grad}_x\,b_0 := J^{\mathrm{e}}(x_0, \mathbf{x})\text{is the electric current density in } K^{\mathrm{e}}.
\end{aligned}$$

More precisely, we find that the functions

$$a_0 = a_0(y_0, \mathbf{y}), \qquad b_0 = b_0(x_0, \mathbf{x}) \tag{6.2}$$

depend only on the variables indicated. This implies that

$$\begin{aligned}
&\frac{\partial}{\partial x_0}a_0(y_0; \mathbf{y}) = 0, \qquad \frac{\partial}{\partial y_0}b_0(x_0, \mathbf{x}) = 0,\\
&\mathrm{grad}_x\,a_0(y_0, \mathbf{y}) = 0, \qquad \mathrm{grad}_y\,b_0(x_0, \mathbf{x}) = 0.
\end{aligned}$$

We also require that the spaces K^{e} and K^{m} be orthogonal with respect to the Minkowski metric $g_{\mu\nu}$. In other words, we require that

$$\langle (x_0, \mathbf{x}), g_{\mu\nu}(y_0, \mathbf{y})\rangle = 0. \tag{6.3}$$

Because of the symmetry of the problem, in K^{e} the fields $\mathbf{a}$ and $\mathbf{b}$ cannot depend on the variables $(y_0, \mathbf{y})$ while in K^{m} they cannot depend on $(x_0, \mathbf{x})$. If we now replace (6.2) in system (6.1), we obtain two systems of Maxwell's equations – one related to the Minkowski space–time K^{e} and the other to K^{m}. If we set $\mathbf{a} := \mathbf{B}$ and $\mathbf{b} := \mathbf{E}$, we obtain the usual Maxwell's equations in K^{e}, while in K^{m} we get

$$\begin{aligned} &\rho^{\mathrm{m}}(y_0, \mathbf{y}) - \operatorname{div}_y \mathbf{B} = 0, && \operatorname{div}_y \mathbf{E} = 0, \\ &\frac{\partial}{\partial y_0} B + \operatorname{curl}_y E = 0, && \frac{\partial}{\partial y_0} E - J^{\mathrm{m}} - \operatorname{curl}_y B = 0, \end{aligned} \tag{6.4}$$

which are Maxwell's equations for magnetic monopoles only. We also have another natural way to split system (6.1). Let K^{e} and K^{m} be the Minkowski space–times with the coordinate systems specified above. We consider now the mixed pairs of variables $(x_0, \mathbf{y})$ and $(y_0, \mathbf{x})$. We now suppose that we deal with a particular symmetric problem in which only the coordinates $(x_0, \mathbf{y})$ are considered while $(y_0, \mathbf{x})$ are neglected. In this way it is easy to derive the Cauchy–Fueter equations from system (6.1). We obtain the same result if we consider such a problem in $(y_0, \mathbf{x})$ coordinates neglecting $(x_0, \mathbf{y})$.

We now note that the operator defined in (5.1) in the case of Maxwell's equations becomes

$$\Delta_{\mathrm{x}} - \frac{\partial^2}{\partial x_0^2} \text{ in } K^{\mathrm{e}} \quad \text{and} \quad \Delta_{\mathrm{y}} - \frac{\partial^2}{\partial y_0^2} \text{ in } K^{\mathrm{m}}. \tag{6.5}$$

These two D'Almbert operators imply that in K^{e} and K^{m}, seen as separated spaces, it is possible to have wave propagation phenomena.

The Cauchy–Fueter operator defined in (5.1) splits in the following Laplace operators:

$$\Delta_{\mathrm{x}} + \frac{\partial^2}{\partial y_0^2} \quad \text{and} \quad \Delta_{\mathrm{y}} + \frac{\partial^2}{\partial x_0^2} \text{ in } K^{\mathrm{e}} \cup K^{\mathrm{m}}, \tag{6.6}$$

whose solutions do not permit wave propagation from K^{e} and K^{m} and vice versa.

We can summarize the above considerations as follows: if we consider separately K^{e} and K^{m} we obtain propagation phenomena, while if we consider an easy symmetric problem related to some coordinates in $K^{\mathrm{e}} \cup K^{\mathrm{m}}$, we obtain that electromagnetic waves cannot propagate.

Remark 6.1. Imaeda [12] deduces Maxwell's equations by making an arbitrary substitution of variables in the Cauchy–Fueter system and gives no physical motivations. In addition to electric monopoles he also deduces the terms related to magnetic monopoles which he subsequently neglects. In this paper we give a natural way to derive Maxwell's equations allowing the existence of magnetic monopoles, but proving that we cannot interact with them by electromagnetic fields.

7. The sheaf of $\mathcal{T}$-regular hyperfunctions

The theory of the so-called quaternionic hyperfunctions in one variable is now a sufficiently developed subject (see [11,16,17]). The main goal of this section is to show that it is possible to develop a similar theory and that this theory has physical interpretations.

In this section, *regular* will mean *left regular*.

Let us start with the following obvious fact whose proof we leave to the reader.

Proposition 7.1. *Let U be any open set in $\mathbb{RH}$. The assignment $U \longrightarrow \mathcal{R}^D(U)$ is a sheaf of right $\mathbb{BH}$-modules.*

Now we prove two basic facts necessary to develop a notion of hyperfunction. The first result is the following.

Proposition 7.2. *Let Ω be a relatively compact set in an open set $U \subset \mathbb{RH}$. Then $H^0(U, U \setminus \Omega; \mathcal{R}^D) = 0$.*

Proof. We have shown in Section 4 that $\mathrm{Ext}^0(M, \Lambda) = 0$. This fact (see [13]) is equivalent to the vanishing of $H^0(U, U \setminus \Omega; \mathcal{R}^D) = 0$. (We recall that this fact implies the analytic continuation property for D-regular functions.) □

Another basic result that we need is the cohomological version of the Mittag–Leffler theorem, whose proof is standard.

Theorem 7.3. *Let $U \subset \mathbb{RH}$ be an open set. Then*

$$H^1(U, \mathcal{R}^D) = 0.$$

Remark 7.4. This last result does not hold if we consider D-regular or $\mathcal{T}$-regular functions of several variables, because Hartog's phenomenon holds, as we have shown in Theorem 4.6. This fact, whose analytic proof is hard to imagine, implies that compact singularities of D-regular and $\mathcal{T}$-regular functions can be removed.

Let Ω be a relatively compact set in an open set U contained in $\mathbb{RH}$. We have the long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(U, U \setminus \Omega; \mathcal{R}^D) \longrightarrow H^0(U; \mathcal{R}^D) \longrightarrow H^0(U \setminus \Omega; \mathcal{R}^D) \\ &\longrightarrow H^1(U, U \setminus \Omega; \mathcal{R}^D) \longrightarrow H^1(U; \mathcal{R}^D) \longrightarrow \cdots \end{aligned}$$

We know that $H^0(U, U \setminus \Omega; \mathcal{R}^D) = 0$ and $H^1(U; \mathcal{R}^D) = 0$. Then we obtain the following isomorphism:

$$H^1(U, U \setminus \Omega; \mathcal{R}^D) \cong \frac{H^0(U \setminus \Omega; \mathcal{R}^D)}{H^0(U; \mathcal{R}^D)}.$$

Let us consider the following sets:

$$\tilde{\mathbb{RH}} = \left\{ X = \sum_{i=0}^{3} \mathbf{e}_i x_i \in \mathbb{RH} \mid x_0 = 0 \right\},$$

$$\mathbb{RH}^+ = \left\{ X = \sum_{i=0}^{3} \mathbf{e}_i x_i \in \mathbb{H} \mid x_0 > 0 \right\},$$

$$\mathbb{RH}^- = \left\{ X = \sum_{i=0}^{3} \mathbf{e}_i x_i \in \mathbb{H} \mid x_0 < 0 \right\},$$

Definition 7.5. Let Ω be an open set in $\tilde{\mathbb{RH}}$ and U an open set in $\mathbb{RH}$ such that Ω is relatively closed in U. Then the right module defined by

$$\mathcal{F}(\Omega) \cong H^1(U, U \setminus \Omega; \mathcal{R}^D) \cong \frac{H^0(U \setminus \Omega; \mathcal{R}^D)}{H^0(U; \mathcal{R}^D)}$$

is called the module of (left) $\mathbb{RH}$-hyperfunctions.

Remark 7.6. Proposition 7.2 and Theorem 7.3 imply that the definition is well defined and does not depend on the open set U.

Theorem 7.7. *The correspondence*

$$\Omega \longrightarrow \mathcal{F}(\Omega)$$

for any open set $\Omega \subset \mathbb{RH}$ *defines a flabby sheaf on* $\mathbb{RH}$.

A simple interpretation of the elements in $\mathcal{F}(\Omega)$ can be given in view of the following result.

Theorem 7.8 (Painlevé). *Let* Ω *be a set in* $\mathbb{RH}$ *for which there exists a null-convex open set* U *in* $\mathbb{BH}$ *such that* Ω *is relatively closed in* U. *Let* $F \in \mathcal{R}^D(U \setminus \Omega)$ *and suppose that* F *is continuous in all of* U. *Then* F *belongs to* $\mathcal{R}^D(U)$, *i.e.* F *defines the zero element in*

$$H^1(U, U \setminus \Omega; \mathcal{R}^D).$$

Proof. As in the classical case, this follows from the application of the Cauchy formula. □

Example. Let $F \in \mathcal{R}^D(\mathbb{RH}^+)$. The function

$$\tilde{F}^+ = \begin{cases} F & \text{on } \mathbb{RH}^+, \\ 0 & \text{on } \mathbb{RH}^- \end{cases}$$

belongs to $\mathcal{R}^D(\mathbb{RH} \setminus \tilde{\mathbb{RH}})$ and defines an element $[\tilde{F}^+]$ in $\mathcal{F}(\mathbb{RH})$ that represents the boundary value of F. Analogously, if $F \in \mathcal{R}^D(\mathbb{RH}^-)$ and

$$\tilde{F}^- = \begin{cases} 0 & \text{on } \mathbb{RH}^+, \\ F & \text{on } \mathbb{RH}^-, \end{cases}$$

then $[\tilde{F}^-] \in \mathcal{F}(\mathbb{RH})$. If $F \in \mathcal{R}(\mathbb{RH} \setminus \tilde{\mathbb{RH}})$, we can write (with obvious meaning of symbols):

$$[F] = [\tilde{F}^+] + [\tilde{F}^-].$$

Remark 7.9. Since a hyperfunction describes the boundary values of two D-regular functions, i.e. of two electromagnetic fields, in our case we have a description of a phenomenon that occurs to electromagnetic fields along a hyperplane of the Minkowski space–time. Suppose that the hyperplane is a nonlinear material which strongly interacts with electromagnetic fields. For example, we can think of a plasma with a particular distribution of momenta or of nonlinear dielectrics. Let us suppose that electromagnetic waves propagate through the hyperplane. Then the hyperfunctions represents the behavior of the fields on the hyperplane.

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